Solving Problems by "Cheating"

Operational Calculi, Function Theory, and Differential Equations

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At first the operators transform the functions, then the functions transform the operators.

William Clifford wrote, in *Common Sense of the Exact Sciences*, that sometimes in algebra we ask a question that has no answer: "If we write down the symbols for the answer to the question in cases where there is no answer and then speak of them as if they meant something, we shall talk nonsense. But the nonsense is not to be thrown away as useless rubbish. We have learned by very long and varied experience that nothing is more valuable than the nonsense we get in this way...We turn the nonsense into sense by giving a new meaning to the words or symbols which shall enable the question to have an answer, that previously had no answer." (Quoted by Alexander Macfarlane "The fundamental principles of algebra," Aug 21, 1898, AAAS).

Say you're in some math course, and you are handed these problems

(1) Solve $y'' + y' - 2y = f$.
(2) Solve $du/dt = Au$, where A is a 5-by-5 matrix with complex entries.
(3a) Solve $u_t = u_x$, given $u(0, x)$.
(3b) Solve $u_{tt} = c^2 u_{xx}$, given $u(0, x)$ and $u_t(0, x)$.
(3c) Solve $u_t(t, x) = u_{xx}(t, x)$ for $t > 0$, given $u(0, x)$.

To solve all of these problems, I will show you a secret two-step method. First, treat mathematical symbols like A or D as if they were numbers, and get "solutions" which are meaningless nonsense

(1) $y = f/(D^2 + D - 2)$.
(2) $u(t) = e^{tA}u(0)$.
(3a) $u(t, x) = e^{tD}u(0, x)$, where $D = d/dx$.
(3b) $u(t, x) = e^{ctD+}f(x) + e^{ctD-}g(x)$, where $D+ = d/dx$, $D- = -d/dx$ and f and g are "arbitrary".
(3c) $u(t, x) = e^{tQ}u(0, x)$ where $Q = (d/dx)^2$.

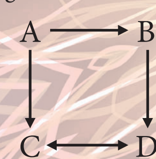
Then figure out the meaning of this meaningless nonsense.

Example 1, sometimes taught in introductory ODE, is usually misattributed to the telephone engineer Oliver Heaviside, but it was written down before Heaviside by a forgotten algebraist, George Peacock. He was a member of the Analytical Society, along with Charles Babbage, and contemporary to William Rowan Hamilton and George Boole. Peacock stated the following "principle or law of the *permanence of equivalent forms*" in his 1830 *Treatise of Algebra*: "Whatever form is Algebraically equivalent to another, when expressed in general symbols, must be true, whatever those symbols denote." This vague principle can be proved as a theorem in several ways. As we shall see, it is a principle that will serve us well.

Example 2 is in a beautiful, seldom taught chapter of linear algebra: "Functions of Matrices".

Examples 3a, 3b, and 3c are "the initial-value problem" for the "transport equation," the "wave equation," and the "heat equation". Such problems are sometimes solved in graduate school as applications of "Banach algebra" or "symmetric linear operator on Hilbert space". Those "modern methods" actually are pumped up versions of the symbolic method, which we are first going to see in matrices and simple ODE's.

Perhaps a good way to illustrate the transmutation idea (and in fact the whole symbolic method) is with a commutative arrow diagram from universal algebra



You start at A with a PDE in a linear vector space. Go across to B with symbolic substitutions for the formulas. Find a way to drop down to D with a meaningful representation by using the symbols in some familiar space such as geometry, algebra, trig, complex plane, matrices, some infinite topological space, semi-groups, Lie algebras. This transformation might have a very complex commutative diagram of its own. Go across to C by finding a solution in the representation domain. Go back from C across to D checking convergence and validity of the solution in the domain. Finally map between the original problem A and the solution form C.

In addition to explaining the meanings of the three symbolic answers given above, we will ponder on the meaning of "meaning".

In conclusion, I will offer a couple of contributions of my own. First there is "the method of transmutations", where I transform the solution of one operator differential equation into the solution of another one. Then there's a trick where I produce a stable, convergent difference scheme of any desired rate of convergence, to any well-posed linear initial-value problem.

Boole's Symbolic Method for Simple ODEs

We start with a venerated old operator calculus, commonly misattributed to the telephone engineer Oliver Heaviside. Basic operations you apply to functions in a calculus course are

- D, differentiating: $f(t)$, a distance function, goes to $f'(t)$, its velocity.
- D^{-1}, anti-differentiating or integrating: $f(t)$ goes to a function $F(t)$ satisfying $F'(t) = f(t)$.
- E_h, shifting: $E_h f(x) = f(x + h)$, where h is a real number, a "parameter" that can be chosen as convenient. Then iterating or repeating, E_h n times gives $f(x + nh)$. Inverting is just shifting back to the left. The two basic operators on functions, D and E_h, are related by a beautiful formula. Taylor's theorem in calculus can be rewritten in operator notation

$$f(x + h) = \sum h^n f^{(n)}(x)/n!,$$

becomes

$$E_h f(x) = \sum (hD)^n f(x)/n!,$$

which we can simplify, dropping the general victim "f", to

$$E_h = \sum (hD)^n/n!$$

Any student of calculus should recognize the right-hand side as the power series of the exponential function with base e and "variable" hD. In brief, $E_h = e^{hD}$ and furthermore $hD = \log(E_h)$. The formula $hD = \log(E_h)$ opens the way to an approximation of D by powers of E_h, the beginning terms of a series expansion of the $\log$. That is to say, a higher-order approximation of differentiation by higher-order finite differences.

These two beautiful formulas connect with and enlighten an important practical problem: discretizing differential operators. Getting good workable approximations using finite differences.

If we move up to functions of 2 or 3 variables, shifting leads straight back into matrix theory. And if we divide by h after shifting by h, we are doing finite differences and almost back to $D = d/dt$. George Boole, most remembered for his logic, wrote a once popular book on the calculus of finite differences.

With that general introduction out of the way, we can turn to Problem 1 above. We will learn an effective elementary technique to solve a linear ordinary differential equation with constant coefficients, of arbitrarily high order. We are given $y'' + y' - 2y = f$. The key step is to abandon Newton's "dot" notation for derivatives, and rewrite the equation in Leibnitz's "D" notation: $D^2y + Dy - 2y = f(t)$, and then to "factor out" the differential operators, thus: $(D^2 + D - 2)y(t) = f(t)$. Then we must dare to write the "solution" in this childish, meaningless way, as $y = f(t)/(D^2 + D - 2)$, or equivalently $y = (D^2 + D - 2)^{-1} f(t)$. Then all we have to do is to figure out the "meaning" of $(D^2 + D - 2)^{-1}$. But if we think of factoring $D^2 + D - 2$ as $(D + 2)(D - 1)$, then

$$1/(D^2 + D - 2) = [1/(D + 2)][1/(D - 1)],$$

or

$$(D^2 + D - 2)^{-1} = (D + 2)^{-1}(D - 1)^{-1}.$$

We will be done if we can just invert $(D + 2)$ and $(D - 1)$, and apply those inverses successively.

But Peacock and Boole and Heaviside showed how to make the solution even simpler, for

$$\begin{aligned} 1/(D^2 + D - 2) &= 1/(D + 2)(D - 1), \\ &= \tfrac{1}{3}[1/(D - 1) - 1/(D + 2)]. \end{aligned}$$

That is

$$(D^2 + D - 2)^{-1} = \tfrac{1}{3}[(D-1)^{-1} - (D+2)^{-1}].$$

We have written the reciprocal of a product as a sum of reciprocals of individual factors, by the method of "partial fractions" which we learned as a useful trick in integral calculus. Now, instead of having to successively invert the factors of $D^2 + D - 2$, we invert them simultaneously, and then add the results. (The very same little trick reappears below, in functional calculus, under the impressive name of First and Second Resolvent Identities.)

So it all comes down to inverting $D - p$, for an arbitrary number p. To invert $D - p$ - to find $(D-p)^{-1}$ - just means, solve the equation $y' - py = f$, for "arbitrary" f. But that's easy! You saw it already in calculus, then again in introductory ODE. The answer is

$$y(x) = e^{px}\left(\int e^{-px} f(x)dx + C\right).$$

where the "arbitrary constant C" remains to be determined by some initial condition. So we rewrite $[1/(D-p)]f$ as

$$e^{px}\left(\int e^{-px} f(x)dx + C\right),$$

and define the inverse, $(D-p)^{-1}$, as

$$(D-p)^{-1}f(t) = Ce^{pt} + e^{pt}\int e^{-pt}f(t)dt.$$

In our example

$$y(t) = \tfrac{1}{3}\left(C_1e^{2t} + e^{2t}\int e^{-2t}f(t)dt + C_2e^{-t} + e^{-t}\int e^{t}f(t)dt\right).$$

In summary

- To invert the product of two operators, multiply the inverses of the factors.
- To invert a composite differential operator such as $(D-p)(D-q)$ invert the factors consecutively, to get the inverse of their composite.

To solve Problem 1, we just wrote $y(t)$ as $(D+2)^{-1}(D-1)^{-1}f$. This is a successive operation with one inverse operator following another. In the same way, any polynomial $p(w)$ with real or complex coefficients can be factored into a product of first-degree factors $(w - w_j)$, and the reciprocal of the polynomial $p(w)$ is the product of the reciprocals, $1/(w - w_j)$. The general problem of solving linear constant coefficient ODEs is thereby reduced to finding the meaning of $[1/(D - w_j)]f$. That is the same as solving the first order ODE $y' - w_jy = f$, and we do that by a simple calculation using integration by parts. The answer from your calculus course is

$$y(t) = e^{pt}\int e^{-pt}f(t)dt + C,$$

(where $p = w_j$). The inverse of the differential polynomial $p(D)$ is the product of the inverses of its factors $D - w_j$. If the roots are distinct, you can expand the composite inverse operator by means of partial fractions, invert each piece, and then just add them up.

But what about a differential polynomial with a complex root? Let's say it's i, to make it simple. Looking back at what we just did, the problem now comes down to a first order DE. with a complex coefficient, $y' + iy = f$. In the solution formula we copied from calculus, the complex number i appears in an exponent. We are back to the exponential function we just struggled with, but with a complex exponent.

You probably already know that $e^{it} = \cos t + i \sin t$. With this interpretation, the inversion we accomplished with real roots also makes sense for complex roots. If a complex root has nonzero real part r, the exponential function in the solution is multiplied by a factor e^{rt}.

A careful logical critic of some steps in this derivation could demand, "What do you mean by an equality between two meaningless expressions?" Peacock's principle of permanence of equivalent forms is a heuristic principle, not a precise theorem. Our answer might be, "We expect in the end to interpret these expressions sensibly, and after such an interpretation, the two sides will really be equal."

For real x, the elementary identity $1/p(x) = \sum c_j/(x - w_j)$, simply "means" that $p(w) \times \sum c_j/(w - w_j) = 1$. So the "meaningless" identity, about a differential polynomial $p(D)$, $1/p(D) = \sum c_j/(D - w_j)$, can be said to simply "mean" $p(D)\sum c_j(D - w_j)^{-1}f = f$. We don't bother to specify appropriate conditions on f, because we can simply "plug this expression into the differential equation" to verify that it really is a solution. (We do not claim to have found ALL solutions.)

Now we move on to Problem 2.

Functions of Matrices

In Problem 2, we are looking at the letter e with the letters tA attached at its upper right corner. That would make sense if the letter A stood for a number. But what entitles us to stick a matrix up there?

We first encounter this exponential notation in elementary school, with a natural number as exponent. It's just the number of times you multiply the base times itself. Then we learn to accept fractional and negative exponents, by requiring the exponential function to satisfy the formula $e^a e^b = e^{a+b}$. Finally, we can even understand an exponent which is a complex number, because the power series expansion of the exponential equals the sum of the series expansion of the cosine plus i times the expansion of the sine.

Oh, that's it! We can just take the power series $e^{ct} = \sum (ct)^n/n!$ which converges for all values of the real or complex variable t, and replace the numerical coefficient c with a matrix coefficient A!

Well, why not? $(tA)^n$ is easy, we know how to multiply A times itself. Matrix multiplication is not the *same* as numerical multiplication, but it obeys all the same axioms, doesn't it? Oh no, the commutative law is false for matrices. But wait! The *powers of a single fixed matrix* do all commute with each other. The n^{th} power times the m^{th} is the $(m+n)^{\text{th}}$ power, in either order of multiplication. So the terms of the power series still make sense if they are $(tA)^n/n!$ instead of $(ct)^n/n!$

But what about convergence? The power series of the numerical exponential converges, because the tail of the series is very small. Even if ct is very big, $(ct)^n/n!$ will be less than the n^{th} power of "epsilon", for epsilon as small as you like, as long as n is bigger than some M (which depends on epsilon). Consequently, the Taylor series of the numerical exponential converges because it is majorized by a convergent power series.

Does this work for the matrix Taylor series?? Yes! We use the beautiful Cayley-Hamilton Theorem: "Any square matrix satisfies its own characteristic function." Meaning, the scalar polynomial $p(x) = \det(A - x)$ becomes the zero matrix when the scalar variable x is replaced by the matrix A. First I will use Cayley-Hamilton, and then I will prove it.

Elementary algebra tells us that if $p(w)$ is a polynomial of degree n, and $m > n$, then w^m can be divided by $p(w)$, with a quotient Q_m and a remainder R_m: $w^m = Q_m(w)p(w) + R_m(w)$. The remainder R_m has degree less than n. By Peacock's principle of the permanence of equivalent forms, we can replace w by A, and write $A^m = Q_m(A)p(A) + R_m(A)$. By Cayley-Hamilton, $p(A) = 0$, so $A^m = R_m(A)$, which is a polynomial in A of degree less than the degree of A. As a consequence, any polynomial in the matrix A can be reduced, term by term, to a polynomial of degree less than n.

To prove the Cayley-Hamilton theorem, remember that any n^{th} degree polynomial in a complex variable z can be written as a product of n linear factors $(z - z_j)$. Therefore, by Peacock's principle again, the characteristic polynomial of A can be factored into a product of n linear factors $(A - z_j)$, $1 \leq j \leq n$. The roots w_j of p are the "eigenvalues" of A. Each factor $(A - w_j I)$ annihilates a corresponding j^{th} eigenvector. If the eigenvalues are distinct, the eigenvectors span the space, and $p(A)$ annihilates every vector in the space, so it must be the zero matrix. (If there is a double root of p, there is a corresponding 2-space that is annihilated, and so on). The proof is complete.

When we write down the exponential function e^{tA}, we expect and require it to satisfy two fundamental identities: $d/dt\ e^{ctA} = cAe^{ctA}$, and $e^{A+B} = e^A e^B$. The validity of these formulas for the matrix exponential, written as power series, follows by a simple argument from their validity term by term for the power series of the scalar exponential. This is part of Peacock's "principle of permanence of equivalent forms." We will spell it out in detail.

Theorem 1: If the function e^{tA} is defined by the power series for the scalar exponential function, then $d/dt\ e^{tA} = Ae^{tA}$. If the two n-by-n matrices A and B satisfy $AB = BA$, then $e^A e^B = e^{A+B}$.
Proof: Since the numerical exponentials e^x and e^y satisfy the two proposed identities, their power series when multiplied term by term or differentiated term-by-term satisfy those identities, that means that the coefficients of the n^{th} power on both sides of the "equals sign" are the same, for all n. But then the two series will be identical if instead of numerical variables x and y, any other symbols are substituted which satisfy the same rules for addition and multiplication.

The reasoning we have used regarding the exponential function works just as well for ANY square matrix, and ANY function represented by a power series (any "analytic function.") The resulting matrix-valued function of matrices will satisfy any identity satisfied by the original analytic function of a real or complex variable. For instance, all of the multitude of trigonometric identities, which originate in reasoning about triangles and circles, continue to be valid for the complex functions $\sin z$ and $\cos z$, and continue to be valid as matrix-valued functions of arbitrary matrices, as long as all the matrices in any formula are mutually commutative (as will be true if they are all functions of one matrix A).

The identities are valid for all z and A, even if the series is convergent only in a bounded region of the z plane. This is an important example of Peacock's principle of permanence of equivalent form.

Complex numbers are included, because complex numbers $z = a + bi$ can be represented by special two-by- two matrices $aI + bJ$, where I is the 2-by-2 identity, and J is

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

This is wonderful magic, and it's too good to be true, at least too good to be true always. The symbolic method can lead to disaster. A bit of caution is required! Multiple-valued functions are dangerous. The very simplest one, the square root, applied to the 2-by-2 identity matrix, takes on not two but four values. (Just put plus-or-minus signs in front of the two 1's in the identity matrix, to display its four square roots.)

Once we feel comfortable with power series of matrices, we naturally ask, what other functions besides the exponential can we compute by power series? The inverse function is a tempting example. The very simplest problem in linear algebra, solving a system of linear equations, is nothing more than finding the inverse of a given square matrix. And the very first power series that you met in school was $1 + x + x^2 + \cdots = 1/(1-x)$. $1/(1-x)$ is the multiplicative inverse of $1-x$, so if we replace x by A, we must get the inverse of the matrix $I - A$. ("I" of course means the n-by-n identity matrix). To check this out, take a very simple example. Choose A as a scalar diagonal matrix, with all entries on the main diagonal equal to 10, and all off-diagonal entries zero. Then $I - A$ is the scalar matrix -9, and its inverse is the scalar matrix $-1/9$. So then, is $-1/9$ the sum of the powers of 10? Does $1 + 10 + 100 + \cdots = -1/9$???? Nonsense! The sum of the powers of 10 diverges. It does not "exist." The supposition that it "exists" is false. It's no surprise that a false conclusion comes from a false hypothesis.

Worse yet, choose as $p(B)$ the identity matrix I. Then the sum of powers $p(x)$ would be just $I + I + I + \cdots$, the sum of copies of I. The n^{th} partial sum would be nI. But $I - X$ is just the zero matrix, and $1/(I - X)$ means "Divide by zero", so of course the series has to diverge.

On the other hand, if the entries in the matrix A are small, then the sum of the powers of A ought to converge. How small should they be? In the scalar case, you probably know that the sum of the powers x^n converges if and only if x is less than 1 in absolute value. What is the analogous statement for the sum of the powers of a matrix A? That is your first homework assignment.

Exercise: For which square matrices of real or complex numbers does the power series $I + A + A^2 + \cdots$ converge?

You will find it hard to answer this without first stating the meaning of "convergence" for a sequence of matrices, and what you would mean by saying a a matrix is "smaller" than the identity matrix. Have fun!

It is tempting to try to carry over to the differential operator D or $p(D)$ the same trick we just saw for inverting (some, not all) matrices. Is D "small" so that the sum of powers of D will converge to $I - D$? No, not at all. Saying D is small, in the appropriate sense here, would mean that the powers of D get small as you take higher powers, which can only mean that for some sufficiently inclusive class of functions f, the sum of f and all its derivatives is small. That works if f is a polynomial, or even an entire analytic function of exponential type. (If you don't know what that means, don't worry about it, just keep on reading). But we don't want to restrict ourselves only to such "nice" functions. The problem itself is meaningful for functions f that are merely differentiable once, or even with sharp corners here and there. We shouldn't need derivatives of order higher than the first. So power series in D is not the right way to make sense of $1/(I - D)$ or $1/(p - D)$. Heaviside's method is the right way to go.

Another Functional Calculus for Matrices

Can we find another interesting class of matrices that all commute with each other? Yes! The diagonal matrices, with zeroes in all the off-diagonal positions. A function $f(A)$ of a diagonal matrix A is easy to write down - just apply f separately to each number on the main diagonal of A. Of course, that function must be well-defined for all the entries on the diagonal of A. For example, we can't define $f(A)$ as A^{-1} if A has a zero on the main diagonal, which is the same thing as saying A has zero as one of its eigenvalues. A matrix with a zero eigenvalue is singular, non-invertible. On the other hand, if A is non-singular, and has an inverse A^{-1}, we can write powers of A^{-1}, which give us the "Laurent series" of A itself. Singularities of $f(z)$ at points in the complex plane other than 0 do not interfere.

This method might seem to be of limited interest, since diagonal matrices are so special. But starting with the diagonal, we can go to a much much bigger class of matrices by the relation of similarity (a term used in a different way in plane geometry.) We say "A is similar to B" if there is a non-singular change of variables that transforms A to B, meaning that there is an invertible matrix P such that $A = P^{-1}BP$. In that case, any function $f(A)$ (a polynomial function or the limit of a convergent sequence of polynomial functions) will be similar to $f(B)$. To see this, first consider just the squares of A and B:

$$\begin{aligned} A^2 &= (P^{-1}BP)(P^{-1}BP), \\ &= (P^{-1}B)(PP^{-1})(BP), \\ &= (P^{-1}B)I(BP) = P^{-1}B^2P, \end{aligned}$$

which means exactly that A^2 is similar to B^2. The same calculation repeated $n-1$ times will show that A^n is similar to B^n. for all n, by the same transformation P and P^{-1}.

Also, if A is similar to C, and B is similar to D by the same transformation P and P^{-1}, then $A+B$ is similar to $C+D$. Therefore by addition, $p(A)$, any polynomial function of A, is similar to $p(B)$. Going over to limits by continuity, we end up being able to apply irregular functions such as step functions, piecewise linear functions, etc, to any B which is similar to some diagonal matrix A.

Symmetric and skew-symmetric matrices turn out to be similar to diagonal ones. The entries on the diagonal matrix A are just the eigenvalues or characteristic values of B, the solutions of the characteristic equation.

So we have two different "functional calculi" for matrices. For any analytic function f, and any square matrix A, the Taylor series of f gives us an interpretation of $f(A)$. On the other hand, if A is similar to a diagonal matrix (A is symmetric, skew-symmetric, "normal", or has n distinct eigenvalues) then all we require from f is to be defined on the diagonal elements of D, which are simply the eigenvalues or the spectrum of A. No conditions of regularity or boundedness on f are needed!

In both of these two functional calculi, any identity satisfied by $p(x)$ is also satisfied by $p(A)$. (Peacock's principle.) It is important to realize that with this "spectral" interpretation of the functional calculus, the seemingly obscure meaning of $f(A)$ for an arbitrary matrix A and any arbitrary function f becomes easy to understand. If A is similar to a diagonal matrix, then by choosing the eigenvectors as coordinates, we see that $f(A)$ operates independently in each coordinate direction, where it is simply multiplication by $f(w_j)$, the function f evaluated at the j^{th} eigenvalue.

The Riesz-Dunford and Laplace-Phillips Functional Calculi

If you have had complex variables, you should remember the Cauchy integral formula. This is the surprising fact that for any analytic function $f(z)$, if you integrate the quotient $f(z)/2\pi i(z-w)$ around a simple closed curve containing the complex number w in the interior, you get the value of w at that point! (If this is news to you, take it on faith for the purpose of this article. The definition of complex integration works in a very natural way, with the usual rules carried over from calculus.)

Now, in this integration formula, nothing prevents us from replacing the complex number w by a matrix A. Then instead of dividing $f(z)$ by $z-w$, we "divide" it by $z-A$. What does that "mean"? Just as in Boole's operator method in Section 1 above, it means' "Multiply $f(z)$ by an inverse operator, $(z-A)^{-1}$". The standard definition of the integral survives under this substitution. We now have a meaningful matrix-valued expression, which is the obvious candidate to serve as the definition of $f(A)$. This formula is called the "Riesz-Dunford functional calculus."

Voila! We have extended the analytic function f from the domain of the complex numbers to the domain of matrices! The set of complex numbers z where $z-A$ is not invertible is called the spectrum of A, or the set of eigenvalues of A. The complement of the spectrum in the complex plane is called "the resolvent set of A", and $(z-A)^{-1}$ is called "the resolvent matrix". It is a matrix-valued analytic function of z, and we can integrate it with respect to z. In the elementary case, the operation A is just multiplication by a fixed complex number w, and we are looking at Cauchy's integral formula. We can multiply the resolvent $(z-w)^{-1}$ by any function that is analytic or holomorphic in the interior of the simple closed curve, and by integrating get $f(w)$. The same integration with respect to complex z, and with a matrix A instead of the complex number w, gives us a definition of $f(A)$ for any analytic function f and any square matrix A. One little detail should worry us. Is the value of this integral independent of the path of integration? In fact, we have to be careful about where we go with respect to the "spectrum," the eigenvalues. In the functional calculus based on diagonalization, we needed to be sure the function f being applied to a matrix A is defined on the spectrum or eigenvalues of A. In our present construction of $f(A)$, we have to be sure all eigenvalues of A are in the interior of our path of integration. With that precaution, Peacock's principle of the permanence of equivalent forms is still good. An identity satisfied by the complex-valued analytic function $f(z)$ is also satisfied by the corresponding matrix-valued function of the matrix A.

Exercise: For e^A defined by the Riesz-Dunford formula, and assuming $AB=BA$, prove $e^{A+B}=e^Ae^B$ and $d/dt\ e^{tA}=Ae^{tA}$.

Since the exponential function is an entire analytic function (has no singularities in the finite complex plane), in this example the path of integration can be taken with arbitrarily large radius.

There are two useful operator identities in the Riesz-Dunford calculus, which are stated in terms of a function $R(z,A)=(z-A)^{-1}$:

- First Resolvent Identity:
$$R(z,A)-R(z,B)=R(z,A)(B-A)\times R(w,A).$$
- Second Resolvent Identity:

$$R(z,A) - R(z,B) = R(z,A)(B-A) \times R(z,B).$$

You might suspect that if the operator is $D = d/dx$, this is just the same as Heaviside's partial fractions formula.

Exercise: Verify this suspicion.

Still More Functional Calculus for Matrices

In Cauchy's formula, a function of two variables, z and w, is integrated with respect to one variable to yield a function of the other variable. In several familiar formulas analogous to Cauchy's, a function of two variables $f(t,w)$ is integrated with respect to t to yield a function $F(w)$. In the Cauchy formula, the function we end up with is the numerator of the quotient being integrated, but evaluated at an interior point. In transforms like the Laplace and Fourier, the result of integration is a new function of the parameter, which is called the Laplace or Fourier transform respectively of the original function that was integrated. Just as in the case of the Cauchy formula and its matrix version, we can replace the scalar parameter in the Laplace or Fourier transform with a matrix A, and use this integral as a definition of the transform function applied to A.

In the Cauchy and Riesz-Dunford calculi, we replace the complex parameter w in the kernel $(z-w)^{-1}$ by the operator A to get the kernel $(z-A)^{-1}$. In the Laplace transform, where the real-valued kernel e^{-wt} is multiplied by an input function $f(t)$ to yield the transform $F(w)$, we can replace the real exponential function e^{-wt} by the operator exponential e^{tA}. The result is an operator-valued function of A, which can and should be called $F(A)$.

In the Riesz-Dunford calculus, the function of a complex variable $f(z)$, known on the boundary of a region, is extended to a point w inside that region merely by use of the special function $1/(z-w)$, the resolvent. The properties of f near w can be deduced from the knowledge of $1/(z-w)$. So if the same formula is used to define f as applied to an operator A, it is merely on the basis of the operator resolvent $(z-A)^{-1}$. It is necessary to justify this by showing that the resolvent of A has all the information about A, just as the fraction $1/(z-w)$ does so for the number w. This is the importance of the two Resolvent Identities. They show that the operator resolvent works just like the complex valued resolvent.

Phillips evidently noticed that just as in the Riesz-Dunford calculus the resolvent of A serves as a building block to define a whole large class of functions of A, so the semi-group generated by A can be used as a building block to define a whole large class of of functions of A.

For example, the Laplace transform of the real-valued exponential function e^{at}, which is the result of integrating from 0 to infinity $e^{at}e^{-wt}$ is $1/(w-a)$. So if we integrate from 0 to infinity the operator-valued function $e^{tA}e^{-wt}$, we get the operator-valued function $(w-A)^{-1}$. Surprise! We have recovered the "resolvent function" of the Riesz-Dunford calculus! The Riesz-Dunford functional calculus is the Laplace transform of the Hille-Phillips functional calculus!

This functional calculus is Chapter 15 of *Functional Analysis and Semi-groups* by Hille and Phillips, where it is credited to Phillips. I knew Phillips when I was an instructor at Stanford University in 1963-64. He collaborated with my adviser, Peter Lax, in their massive theory of scattering. A few years later I worked with Einar Hille, when he was at the University of New Mexico after retiring from Yale and before moving on to the University of California in La Jolla.

Nelson Dunford was a professor at Yale University. With Jack Schwartz he co-authored their three-volume bible, *Linear Operators*. Schwartz was one of my professors at NYU in the 1950s. *Functional Analysis* by Frigyes Riesz and his pupil Bela Sz.-Nagy was our textbook. Riesz was one of the preeminent founders of functional analysis. He spent most of his life at the University of Szeged in Hungary, because in the University of Budapest math department Leopold Fejer was already present, and "one" was the maximum number of Jews allowed there. In 1988 on a visit to Szeged I had lunch with Nagy, who was then 75 years old. Riesz was no longer alive. Nagy told me that Riesz had survived the Holocaust by hiding in his apartment, where friends brought him food.

Symbolic Solutions of Partial Differential Equations

Now we must turn to numbers 3a, b, and c, in our imaginary test questions. Any linear initial-value problem—the heat equation, the wave equation, the Schrodinger equation - are the most familiar - can be represented as $du/dt = Au$, with appropriate choice of A, and so can be solved symbolically by $u(t) = e^{tA}u(0)$. What is the meaning of e^{tA}?

Problem (3a), $du/dt = c\, du/dx$, is the simplest and easiest. If we represent spatial differentiation by the symbol D, this PDE becomes $du/dt = cDu$. Returning to our naive innocent mode, we would just write down $u(t,x) = e^{ctD}u_0(x)$. But in Section 1 we met Boole's formula, which informs us that $e^{hD}f(x) = E_h f(x) = f(x+h)$. The group of operators generated by cD is just shifting to the right and left at speed c, $u(t,x) = u(0, x+ct)$.

But notice a striking discrepancy. These solution operators can operate on any f, even one with sharp corners or jumps (a step function.) Such irregular functions cannot satisfy any differential equation. Can we find a way to make sense of differentiating non-differentiable functions? (Hint: Yes. Instead of classical functions, use generalized functions, also known as Schwartz distributions.)

We can just as well write down the PDE $du/dt = -c\ du/dx$, with solution $u(t,x) = u(0, x-ct)$. If we take second derivatives, we get the same second-order pde for both shifts, Problem (3b), $u_{tt} = c^2 u_{xx}$. This is the one-dimensional wave equation, which describes signals propagating along a string or a wire at speed c. In operator notation, $u_{tt} = c^2 D^2 u$ or, using operators D_t and D_x, $(D_t^2 - c^2 D_x^2)u = 0$. With operators just as with numbers, the difference of two squares is easily factored, to $(D_t + cD_x)(D_t - cD_x)u = 0$.

This factorization can be rewritten in the opposite order. Each factor annihilates its corresponding shift operator. $(D_t + cD_x)$ annihilates any function of the form $f(x-ct)$, and $(D_t - cD_x)$ annihilates any function of the form $f(x+ct)$, so any linear combination of a function of $x+ct$ and a function of $x-ct$ is annihilated by the product of the two operators. A general solution requires two arbitrary functions f and g, $u(t,x) = f(x+ct) + g(x-ct)$. By appropriate choice of f and g, we can satisfy two initial conditions, for both the initial value of $u(t,x)$ itself at time $t=0$, and the initial value of its derivative u_t.

Exercise: Now consider the Cauchy data $u(0,x) = g(x)$, $u_t(0,x) = h(x)$. Derive d'Alembert's formula

$$u(t,x) = \frac{1}{2}\{g(x-ct) + g(x+ct)\} + \frac{1}{2c}\int_{x-ct}^{x+ct} h(\xi)d\xi.$$

The initial value, $u(0,x)$ sends half of itself to the right and half to the left, at speed c. To this is added the integral of the initial velocity $u_t(0,x)$, also sent to the right and left at speed c.

In Problem (3c) we meet another important elementary pde of evolution, the "heat equation" $u_t = u_{xx}$, which has the symbolic solution $u(t,x) = \exp(tD_x^2)$. We will obtain a solution valid for all real x, and for all positive time. It will model heat flow from an instantaneous heat pulse concentrated at the origin $x=0$ at time $t=0$.

You might know that the Fourier transform turns the space derivative D into multiplication by a new space variable. So to interpret $\exp(D^2)$.we might first consider $\exp(x^2)$. But this unpleasant function has explosive growth. Better try instead $\exp(-x^2)$. This is nice. It is a multiple of the familiar bell curve or normal curve from statistics. One interpretation of the heat equation (or "diffusion equation" as it also called) is an evolution of bell curves, starting with a singular peak at zero, and flattening out more and more, eventually going to zero everywhere. In other words, the variance goes from zero to infinity as time goes from zero to infinity. We can accomplish this by putting a factor $1/ct$ into the exponent along with $-x^2$. (The coefficient c lets us adjust the rate of decay to fit in with the given differential equation.)

Thus we have a trial function $\exp(-x^2/ct)$. This expression goes to 1 as t goes to infinity, pointwise for all x. As t goes to zero, for any x not equal to zero, it goes to zero, with exponential rapidity, but for $x=0$, it is constant $=1$, independent of time. Physical reasoning tells us that a positive amount of heat energy initially concentrated at a single point would have to be very, very hot. Much more than 1. Also, at all values of x, the temperature should eventually fall to zero, as the heat energy moves away to the right and left. So we need a correction factor, some negative power of t that gets really big as t goes to zero, and gets small as t goes to infinity. Not knowing in advance what is the right negative power, we can just call it a, and now have as our trial solution $u = t^a \exp(-x^2/ct)$. "Plug this in" to the differential equation, calculate u_t and u_{xx}, set them equal and solve for a and c. Sure enough, you will get $c=4$ and $a = -1/2$.

This checks out as a solution to Problem (3c). But you may object, "There are many other solutions of this equation. What is so special about this one?" In fact, this is often called "the fundamental solution." To see why, suppose you are given some arbitrary function $f(x)$ as the initial temperature. What is the temperature $u(t,x)$ at future times? The equation is linear, so we can add solutions to get new solutions. Think of this initial temperature $f(x)$ as the "sum" of separate contributions, which are each concentrated at a single point, and zero elsewhere. The solution corresponding to each one of these isolated initial temperatures is the fundamental solution we have just found, shifted over to the particular point in question, and multiplied by an appropriate factor, according to the value of the initial temperature $f(x)$ assigned at that point. The "sum" of all these separate pointwise contributions to the initial temperature is the integral of the product of the initial temperature $f(x)$ times the shifted fundamental solution we have just obtained. This integral of a shifted product is called the "convolution" of the two functions. Thus the special solution we have just derived is the seed from which all other solutions of the initial value problem are expressed as convolution integrals.

We have solved all three of the last group of test problems, which came from from the advanced subject of linear partial differential equations. Each of the differential operators appearing in these initial-value problems is the "generator" of a "semi-group" of "solution operators", which we represented symbolically as the exponential function of the generator. The generator is a differential operator. It is the derivative of the solution operator. These generators and semi-groups operate on infinite-dimensional function spaces.
Some linear partial differential equations are useful in physics. To study them, we must move up to infinite dimensional spaces. That may sound intimidating. But after all, we have been talking about n-by-n matrices without restricting n. If n is finite but "really really big", wouldn't that be close to infinite dimensions? To put it into language more acceptable in the classroom, can we do infinite dimensions by approximating from finite dimensions? Just as we approximate π by a finite decimal expansion, carried out as far as necessary.

In working with infinite-dimensional operators, we must make new distinctions which did not appear in the finite-dimensional world of matrix theory. We must distinguish between bounded and unbounded operators. The differential operators are unbounded. As such an operator is iterated, it requires a more and more restricted function space on which to operate - smoother and smoother functions. The solution operators, in contrast, are bounded. This is already evident back in elementary calculus, where differentiation is unbounded, but integration is bounded.

The axiomatic study of algebras of bounded operators is included in the highly developed theory of "Banach Algebras". This subject is largely the work of the famous Israel Moiseyevich Gelfand and his collaborators. (They used the name "normed rings" for their theory, which later became "Banach algebras.") The different "functional calculi", including infinite series, which we developed for matrix algebras, are true more generally, for various other Banach algebras.

In order to understand unbounded operators, such as the differential operators associated to initial and boundary value problems, it is necessary to make additional hypotheses, such as symmetry, or having compact inverses. The existence of a bounded semi-group - that is, the solvability of the initial value problem - is a powerful tool. The Hille-Phillips functional calculus which we presented for matrices is available in the infinite dimensional context. So is the Riesz-Dunford calculus, based on Cauchy's integral formula. Diagonalization works for the important special class of symmetric compact operators.

The spectra of unbounded operators can be more difficult to manage. No longer discrete sets of eigenvalues. But most of the tools we met in studying matrices are major weapons in dealing with unbounded operators on infinite dimensional spaces. Diagonalization, and the Riesz-Dunford and Hille-Phillips representations, are chapters in these advanced topics. The facts that were presented in Section 2, on functions of matrices, carry over, with due precautions, to the infinite-dimensional function spaces of linear partial differential equations. Infinite series works for bounded operators, in Banach algebras.

Final Methodological Musings

We have surveyed a handful of different problems, each of which is usually expounded in isolation from other examples, and hopefully we have learned to see the operational method in great generality. This honest work entitles us to enter the realm of philosophical musing, and ask "What are we talking about, when we say certain formulas are meaningless, and others are meaningful or legitimate?"

If Heaviside's formulas about D seem "meaningless", then to a child in elementary school, the associative laws of addition and multiplication also seem meaningless. To the naive beginner, it is the nouns, the numbers, that are "real", while the verbs, the operations, are meaningful only when in actual operation. As Anna Sfard has emphasized, reification, the move from a verb to a noun, is intrinsically difficult. Students complain that the reification of a verb is "too abstract."

When, with Oliver Heaviside, we write down a meaningless "fraction," $1/(2-D)$, or when,with a 5-by-5 array of numbers called A, we write down e^{tA}, we are attempting to extend the domain of definition of division or exponentiation And we want the extended function to still be "the same" function in the new domain, to be "just like it was" in the old domain. What does that "mean"? It means Peacock's principle of "equivalence of permanent forms". The extended function should still satisfy "the same" conditions and formulas. For the reciprocal function, for example, it means that "the reciprocal of a product is the product of the reciprocals". For the exponential, it means that "the exponential of a sum is the product of the exponentials". Earlier in your schooling, you went through this same process of enlarging a domain. In the 3rd or 4th grade you started to "learn fractions." It's easy to understand adding fractions. However, you also have to multiply them!

Dividing by 2 becomes "multiplying" by ½. Until now, multiplying was just multiplying by a natural number, which meant adding the same thing up a certain number of times. Multiplying by a fraction? That doesn't "mean anything". Yes of course, we say "half of six is three", but now we change that to "one half times six equals three." How does "of" get changed to "times"?

These worries are not discussed in class. Just copy the teacher and the textbook, until you get used to it. Looking back on that now, we see that the "times" operation, multiplication was being extended from an old domain, the natural numbers, to a new one, the rational numbers. The rules for multiplying by fractions are compelled to be what they are, in order to preserve the existing structure, to make the rules for natural numbers continue to be valid for fractions. Peacock's principle is not a lucky surprise, it is the guiding principle by which we carry out the extension.

When we meet matrix multiplication, it is certainly clear that the "product" of two square matrices is not the same thing as ordinary numerical multiplication. By what right do we call it by the same name, "times" or product? In other words, how do we know that it is an extension of ordinary multiplication, that it follows all the same rules? In fact, at the beginning of the linear algebra course, it is necessary to check that matrix multiplication is associative, and distributive (not commutative).

The process of extending domains is overt in a complex variables course. The zeta function is first defined by a series convergent in the right half- plane, and then extended by means of a functional equation that the function satisfies. When we extended the exponential function to the solution operator of a well posed initial-value problem, we did have to prove that this extended function satisfies the multiplicative property of the exponential. Fortunately, the proof is just a few lines, given the status of the solution operator.

Theorem 2: Let $P[f_1(x), \cdots, f_n(x)]$, be a polynomial in the n variables $f_j(z)$ where $f_j(z)$ are analytic functions of z in some domain that contains some interval $a \le x \le b$ of the x axis. If on the interval those functions satisfy an identity $P[f_1(x), \cdots, f_n(x)] = 0, \quad a \le x \le b$, then for all z in the domain it is true that $P[f_1(z), \cdots, f_n(z)] = 0$. (From Churchill, *Complex variables and applications*, McGraw Hill, 1966, page 262.)

While this theorem seems to be quite general, and suffices for the examples in Churchill's book, we need to also allow division in order to include the resolvent function. But in order to include division, we must impose an appropriate condition to avoid dividing by zero. Not only multiplication and division of functions, but also composition of functions observes permanence. But the principle breaks down for square roots and other multi-valued functions.

In elementary ODEs, we freely write $p(D)$ to mean a polynomial with real coefficients, with, as a "variable", the differentiation operator. "Multiplication" means successive application of a linear operator. We have extended the domain of the polynomial function from numbers to differential operators, because the differential operators follow the same rules as numbers. There is an amusing reversal of role here!

Differential operators are things that operate on functions. D doesn't "mean" anything by itself, it only "means" something when you give it a function f to operate on, Df. On the other hand, for some f's, in particular for polynomial functions $p(x)$, we also write down and understand the result of doing it in the opposite order. $p(\cdot)$ transforms one operator D to another one, $p(D)$. Instead of operators operating on functions, we have operators being operated on by functions.

Two of my own Papers are in this Symbolic Method!

In conclusion, I make a confession. In writing this very article, I noticed for the first time that in the 1970's I employed the very same "operator calculus" or "symbolic method" that I am now preaching.

Two articles are superficially separate and unrelated. One is called "The Method of Transmutations." It shows how to use the solution of one operator problem to get the solution to another problem involving the same operator The second article, co-authored with Tosio Kato, introduced an infinite number of highly accurate approximation methods for linear initial-value problems.. These formulas are now called "rational approximations to semi-groups." We used rational approximations to the complex-valued exponential in order to approximate the operator-valued exponential ("strongly continuous semi-group").

A "transmutation" is a transform that takes the solution to one initial or boundary value problem involving an operator A, and yields the solution to a different problem involving that operator. For instance, we might have $u_t = Au$ and $v^2 = A^2 v$ or $u_{tt} + au_t + bu = A^2 u$, and $\epsilon v_{tt} + av_t = Av$.

This transmutation article is not a normal math research publication. There is no theorem. Nothing is claimed to be proved! A collection of formulas, never previously brought together, are shown to be examples of the transmutation procedure. These examples show you how to find a transmutation formula. Just take away the operator A, and replace it with multiplication by a real or complex number w. If, as in all these examples, the solutions to these two "concrete" problems can be related by a known transform formula, voila! Just plug in your operator A in place of w, and you have your transmutation! (This is what we did up above in Section 2 to define an analytic function of a matrix—we "plugged in" the matrix for the real or complex number in the standard formula.) Transmutation is a very broad generalization of the usual applications of the Fourier or Laplace transform.

A transmutation may turn a singular perturbation problem into a regular one. Or Problem I could be a complicated concrete problem from an applied area, while Problem II could be a standard problem solved in every introductory text book. Problem I could involve confidential or secret data or information, or empirical date known only approximately. The difficulties or peculiarities of a singular or empirical operator-theoretical problem may be taken apart. with the singularities brought down to real or complex analysis, and the operator-theoretic complications shifted over to a known standard problem. Carroll and Showalter used this transmutation method to study operator equations generalizing the singular Darboux equation that governs spherical means.

My second self-reference, the one about "rational approximation to a semi-group", came about simply by noticing that the standard approximation schemes--the first-order implicit scheme used by Hille and Kato, and the second-order scheme called "Crank-Nicholson"--were both merely the simplest examples of approximating the exponential function by rational functions - ratios of polynomials - rather than purely by polynomials, as in Taylor series. Contemplating Hille and Crank-Nicholson, I wondered, "Why stop there?" In fact, there is no barrier to cubic or quartic accuracy, or as high as you like!

At first it may seem laborious to divide by a polynomial. But No! Heaviside's elementary trick of partial fractions saves the day. Just rewrite the polynomial divisor as a sum of inverses of linear functions. Values of the "resolvent function", in other words.

The ratios of polynomials, when applied to the first-difference operator, become combinations of forward and backward differences, also called "implicit" and "explicit" differences. The use of implicit or forward differences corresponds to division instead of multiplication by the basic difference operator. This is necessary in order to keep the iterations bounded. This necessary condition is called "stability" in approximation theory.

For polynomials of degree n in the numerator and degree m in the denominator, the best possible approximation to the exponential function is of order n + m, and is given by the Padé table. I used the Riesz-Dunford calculus for the proof, but Tosio Kato was able to get sharper estimates using the Hille-Phillips operational Laplace transform that I discussed above in Section 2.

Thomée and Brenner then proved an optimal estimate conjectured by Kato. More recently, Patricio Jara and others at Louisiana State University extended these results in several significant ways.

A worked-out computation offered in Jara (2008) is startling in its orders-of-magnitude superiority over the standard method. Specifically, the reader is referred to Figure 2 of this paper; his graphs of the comparative errors. You will notice that the scales of the two graphs differ by 5 orders of magnitude!

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References

Bade WG (1953) An Operational Calculus for Operators with Spectrum in a Strip. Pacific J Math 3:257-90.

Berg EJ (1936) Heaviside's Operational Calculus as Applied to Engineering and Physics. Electrical Engineering Texts.

Boole G (1872) Calculus of Finite Differences.

Churchill R (1966) Complex variables and applications. McGraw Hill, pp 262.

Clifford W (1885) Common Sense of the Exact Sciences. Appleton.

Friedman F (1991) Lectures on Applications-Oriented Mathematics. Dover, Mar 27.

Gelfand I, Raikov D, Shilov G (1946) Commutative Normed Rings. Uspeki Mat Nauk 48-146.

Goldstein JA (1985) Semigroups of Linear Operators and Applications. Oxford University Press, Oxford.

Hersh R (2015) Peter Lax, Mathematician, An Illustrated Memoir. American Mathematical Society.

Hersh R (2014) Experiencing Mathematics, What Do We Do When We Do Mathematics? American Mathematical Society.

Hersh R, Kato T (1979) High Accuracy Stable Difference Schemes for Well-Posed Initial Value Problems. SIAM J Numer Anal, pp 670-82.

Hersh R (1975) The Method of Transmutations. Lecture Notes in Mathematics, Partial Differential Equations and Related Topics. Springer-Verlag, no 446, pp 264-82.

Hille E, Phillips R (1957) Functional Analysis and Semi-groups. American Mathematical Society.

Jara P (2009) Rational approximation schemes for solutions of the first and second order Cauchy problem. Proc Amer Math Soc 137:3885-98.

Jara P (2008) Rational approximation schemes for bi-continuous semigroups. J Math Anal Appl 344(2):956-68.

Kato T (1980) Perturbation Theory of Linear Operators. Springer.

Lax PD (1997) Linear Algebra. John Wiley & Sons.

Lax PD (2002) Functional Analysis. Wiley-InterScience.

Macfarlane A (1898) The fundamental principles of algebra. AAAS, Aug 21.

Peacock P (1830) A Treatise of Algebra. Cambridge, Deighton.

Riesz F, Sz-Nagy B (1955) Functional Analysis. Frederick Ungar Publishing Company.

Sfard A (1992) Operational origins of mathematical objects and the quandary of reification - the case of function. In G Harel & E Dubinsky (eds) The Concept of Function, Aspects of Epistemology and Pedagogy, MAA Notes 25. Mathematical Association of America, pp 59-84.

Sfard A (2008) Thinking as Communicating. Cambridge University Press, New York.