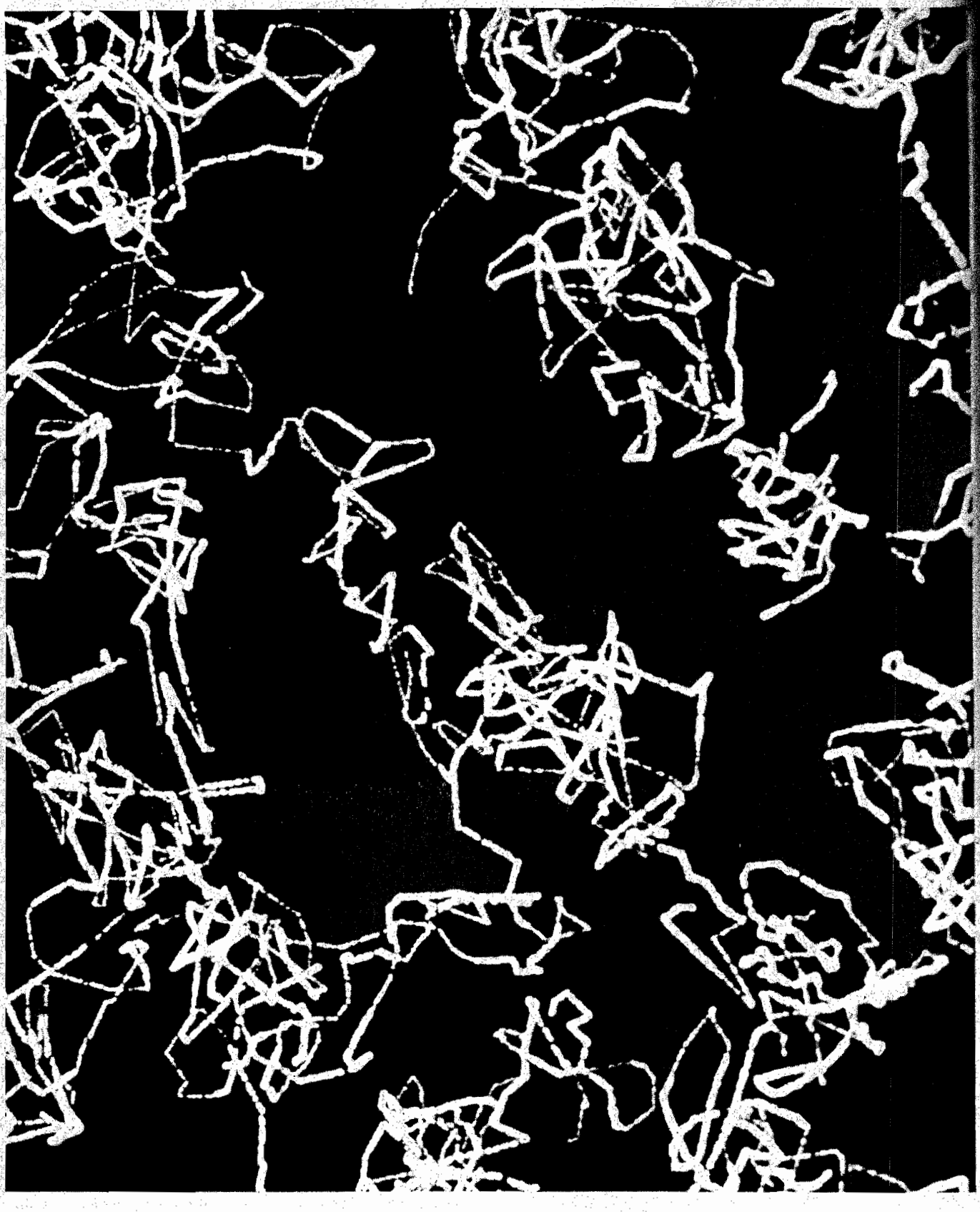


COMPUTER "SNAPSHOT" of the paths of a number of molecules in a hypothetical fluid symbolizes one of the two main themes of this article: the mathematical analysis of the Brownian motion of a small particle buffeted by a turbulent "sea" of such molecules. In order to make the photograph, each imaginary molecule was represented by a bright dot on the face of a cathode ray tube. By focusing a camera on the screen and leaving its shutter open, it was possible to record the trajectories of the moving dots on photographic film. The molecular-dynamical calculations of the paths were fed into the display system from a computer, in which the location of each molecule was represented by a set of three numbers, specifying the three-dimensional coordinates of its center. The molecules were "shaken" one by one, using a mathematical technique called the Monte Carlo method; according to this technique, a particle chosen at random was displaced by an amount that was determined by picking one of a series of random numbers generated by the computer. Various boundary conditions can then be simulated by making some moves "legal," and others "illegal." The experiment was performed by B. J. Alder and Thomas E. Wainwright at the Lawrence Radiation Laboratory of the University of California at Livermore.

Brownian Motion and Potential Theory

The discovery that these two apparently unrelated branches of physics are in some sense mathematically equivalent has led to a new subject known as probabilistic potential theory

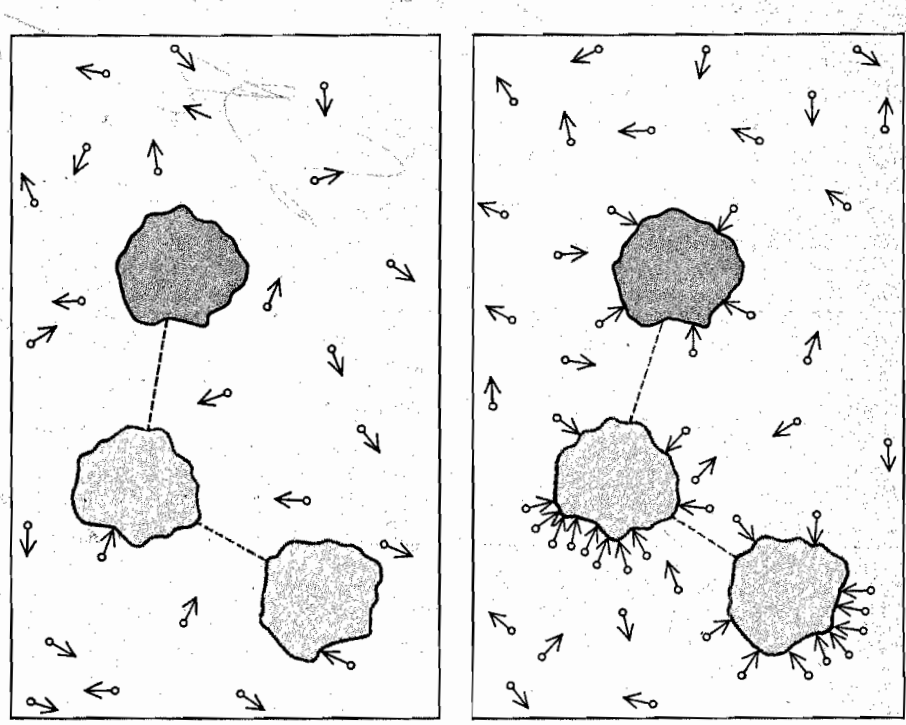
by Reuben Hersh and Richard J. Griego

One of the most exciting events of the past decade or so in the field of mathematical analysis has been the appearance of a new subject called probabilistic potential theory. In essence this subject represents the marriage of two major branches of theoretical physics: the probabilistic theory of random processes, which studies such phenomena as the Brownian motion of a small particle buffeted by a turbulent "sea" of molecules, and potential theory, which studies the equilibrium states of a homogeneous medium, for example the distribution of heat in a solid body at thermal equilibrium. The development of probabilistic potential theory is rooted in the discovery that these two apparently unrelated branches of physics are in some sense mathematically equivalent; in other words, the mathematics of one can be translated meaningfully into the mathematics of the other. From this remarkable circumstance have flowed many unexpected insights into both subjects. Before pointing to the accomplishments of probabilistic potential theory, however, it will be helpful to review briefly the two separate lines of inquiry that the new theory connects, namely the theory of Brownian motion and potential theory.

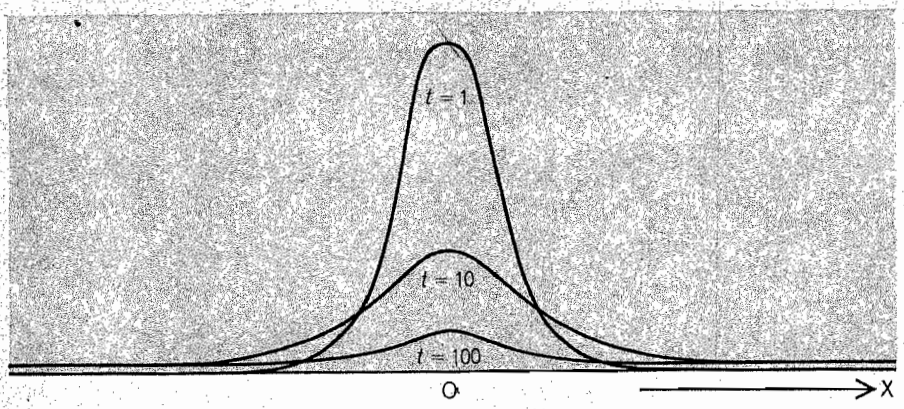
The phenomenon of Brownian motion was described in 1827 by the Scottish botanist Robert Brown in the course of an investigation of the fertilization process in a newly discovered species of flower. Brown observed under the microscope that when the pollen grains from the flower were suspended in water, they performed a "rapid oscillatory motion." At first he believed this motion was peculiar to the male sexual cells of plants. He soon found, however, that particles of other organic substances, bits of petrified wood and even chips of glass or granite exhibited the same motion.

It was not until the 1860's that the problem of the cause of Brownian motion really began to bother theoretical physicists. Early attempts to explain the phenomenon in terms of fluid currents in the host medium had had to be rejected because observation showed that the motions of two neighboring particles seemed quite uncorrelated with each other. Other properties of Brownian motion were equally intriguing; for example, the higher the temperature the faster the motion, the smaller the particle the faster the motion, the more viscous the medium the slower the motion. In addition, the given particle appeared equally likely to go in any direction, and the past motion of a particle seemed to have no bearing on its future motion. Last but not least, the motion never stops.

The relation between temperature and speed seemed to suggest a molecular origin for the Brownian motion, since according to the kinetic theory of heat the temperature of any substance is proportional to the average of the square of the speed of its molecules. Thus a higher



MOLECULAR ORIGIN of Brownian motion was originally suggested by the observation that Brownian particles move faster at higher temperatures; according to the kinetic theory of heat higher temperature simply means more rapid molecular motion. Even the smallest particle observable through a microscope, however, is far too large to show observable motion as a result of a series of kicks from one molecule at a time (*left*). Instead it has been shown that the random motion of a Brownian particle is caused by the random discrepancy between the molecular pressures on different surfaces of the particle (*right*).



BELL-SHAPED CURVES result when one plots the successive positions of a Brownian particle on a graph. The horizontal axis of the graph measures the distance traveled in a given direction, assuming that at a time $t = 0$ the particle is at point $x = 0$. The vertical axis measures the probability that the particle will be at any given point at time $t = 1$, 10 and 100 seconds respectively for the three curves. Such symmetrical curves are called normal or Gaussian densities and represent the quantity $e^{-x^2/4Dt}/2\sqrt{\pi Dt}$, where e is the base (equal to 2.7182818) of a natural system of logarithms and D is the diffusivity of the medium.

temperature means more rapid average molecular motion, and observation had already shown that higher temperature means more rapid Brownian motion. On the other hand, any simpleminded notion that the jerky movement of Brown's particles was due merely to kicks from single molecules was out of the question. Even the smallest particle observable in a microscope is far too large to show observable motion as a result of a series of kicks from one molecule at a time.

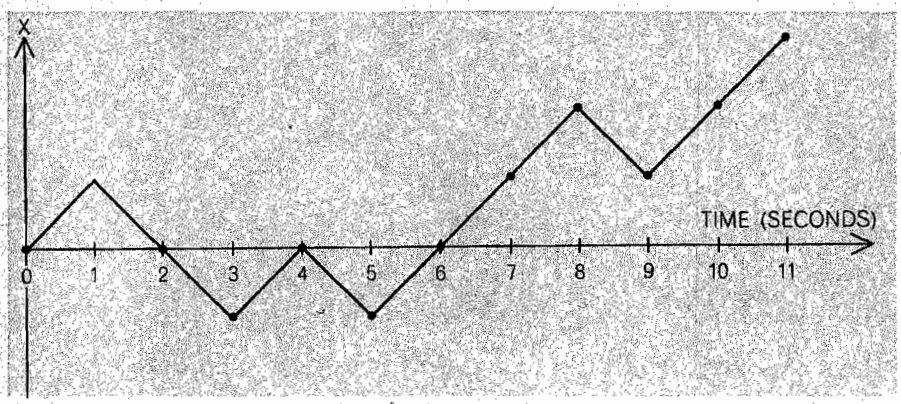
A major advance came in the form of a theoretical analysis in statistical mechanics completed by Albert Einstein in 1905, the year he published his first paper on relativity. At the time many leading physical scientists, including Wilhelm Ostwald and Ernst Mach, regarded molecules and atoms not as real entities but as intellectual figments that might be useful for explaining certain natural phenomena. (This view, which is hardly remembered today, is somewhat reminiscent of the way many physicists later regarded the wave-particle duality of quantum mechanics.) Einstein, reasoning on the basis of the kinetic theory of heat, determined that if an observable particle were in the midst of a molecular bombardment, it would then describe a random motion caused by the difference in the number of blows it might receive at any instant on, say, its left and right surfaces. The smaller the particle, the more likely it would be that this difference would be sufficient to cause a detectable push. The less viscous the fluid, the faster and farther the particle would go as a result of each push. Once each tiny step was stopped by the fluid's viscosity, any future motion would depend only on the random discrepancy between molecular pressures on the particle's left and right surfaces, its front and back surfaces or its top and bottom surfaces.

Over a period of time such a hypothetical particle would tend to drift from its original position. Its exact position after a certain number of seconds is of course unpredictable, but it turns out that one can state in a general way where the particle is likely to be. If the experiment is performed many times and the successive positions of the particle are plotted on a graph, one obtains a bell-shaped curve such as the curves shown in the top illustration on this page. The horizontal axis in each case measures the distance traveled in any given direction, say left or right, assuming that at time $t = 0$ the particle is at the point $x = 0$. The vertical axis of the graph measures the probability that the particle will be at any given point at time $t = 1$, 10 and 100 seconds respectively for the three curves.

It is evident from such graphs that the most probable position is always the original position and that the farther away a position is, the less likely the particle is to be there at any given time. Moreover, the graphs are symmetrical since the movement of the particle is unbiased between left and right. As one might expect, the three curves show that the longer the particle drifts, the likelier it is to wander from its starting point. Bell-shaped curves such as these are called normal or Gaussian distributions and they typically arise in situations where the measured quantity is the sum of a great many independent but essentially identical random variables, in this case the many little pushes that add up to the total motion.

It is remarkable and amusing that only after Einstein had completed his calculations did he learn that the phenomenon he was predicting was already well known! He wrote later: "My major aim in this was to find facts which would guarantee as much as possible the existence of atoms of definite finite size. In the midst of this I discovered that, according to atomistic theory, there would have to be a movement of suspended microscopic particles open to observation, without knowing that observations concerning the Brownian motion were already long familiar."

It was largely this work of Einstein's that finally put out of fashion the view that molecules and atoms might be fictitious. In 1926 Jean Perrin received the Nobel prize for an experimental application in 1909 of Einstein's results. By ob-



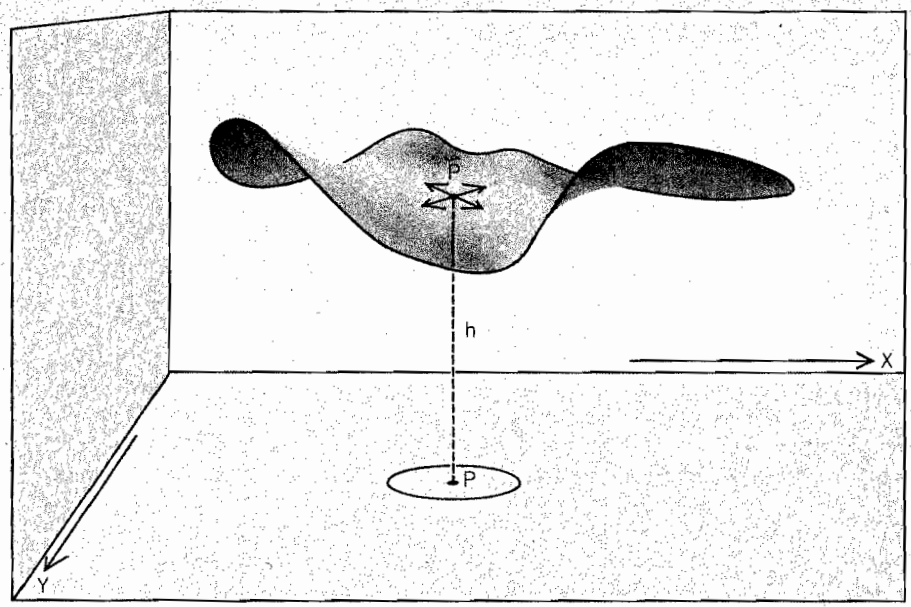
INTUITIVE NOTION of what is involved in Norbert Wiener's mathematical model of Brownian motion can be obtained from this illustration. The black zigzag line is the path traced out in a finite time period (in this case 11 seconds) by a one-dimensional motion that changes direction only at the instants $t = 1$, $t = 2$ and so on. If this process were continued for an hour, say, there would be only a finite number of paths ($2^{3,600}$, to be exact), and one could say that the particle chooses one path at random in the sense that each path has a probability of $1/2^{3,600}$. Such a process is called "random walk" or "drunkard's walk." Wiener solved the problem of going to the limit of infinitely small time increments.

serving actual Brownian movement Perrin was able to measure the predicted displacements and was thereby able to compute the diffusivity parameter D. According to Einstein's formula, $D = 2RT/Nf$, where R is a universal constant, T is the absolute temperature, N is Avogadro's number (the number of molecules in a gram-molecular weight of a gas) and f is a viscosity coefficient. In this way Perrin succeeded in obtaining a value for Avogadro's number, one of the fundamental constants of nature: he found that N is approximately equal to 6×10^{23}.

As far as physicists were concerned, Einstein's and Perrin's investigations left the problem of Brownian motion in reasonably good shape; more recent work has tried to refine and justify Einstein's calculations on the basis of the general Maxwell-Boltzmann equations of statistical mechanics. For mathematicians, however, the story does not really begin until 1920, when Norbert Wiener wrote his first paper on Brownian motion.

Whereas from a physical viewpoint Einstein's calculations and Perrin's experiments had explained Brownian motion quite adequately, from a mathematical viewpoint the subject was still tantalizingly confused. The heart of the difficulty was to make precise mathematical sense out of the notion of a particle moving "at random." Everyone knows what it means to pick between heads and tails at random; it means each alternative has a probability of 1/2 (if we assume that the toss is fair). The Brownian particle follows a path that is in some sense chosen at random from among all possible paths. The set of all possible paths, however, is a very large and complicated one, and it was one of Wiener's major achievements in mathematics to show in what sense one can speak about choosing from this set at random.

We shall not attempt here even to summarize Wiener's argument. Nonetheless, an intuitive notion of what is involved can be obtained by considering the path traced out in a finite time period (say an hour) by a one-dimensional motion, which changes direction only at the instants $t =$ one second, $t =$ two seconds and so on [*see bottom illustration on opposite page*]. In this case there are only a finite number of possible paths ($2^{3,600}$, to be exact), and one could say that the Brownian particle chooses one path at random in the sense that each path has a probability of $1/2^{3,600}$. Such a process, made up of discrete steps, is sometimes called "random walk" or "drunkard's walk." The difficulty is in going to the limit of infinitely small time increments.



ELASTIC MEMBRANE is stretched across a stiff, closed frame that is twisted into some fixed shape in space in this illustration of the role of harmonic functions in potential theory. The configuration of such a membrane is given by the height h of each point $\tilde{P}$ on the surface of the membrane. Directly below each $\tilde{P}$ on the membrane is a point P on the base plane, which has coordinates x, y. Besides being continuous, the function $h(x, y)$ has the following simple property: If P is a point in the x, y plane, and Γ is a small circle with its center at P, then the value of h at P (that is, the height of the membrane above P) equals the average of the values of h for all points on the circle Γ. This is called the mean-value property, and a continuous function h possessing this property is called a harmonic function. In this case the position of $\tilde{P}$ (the point on the membrane above P) is determined by the sum of the tension forces exerted on $\tilde{P}$ by the surrounding portion of the membrane (*arrows*). If the membrane is in equilibrium, these forces must cancel, so that the number of nearby elevations greater than that of $\tilde{P}$ must be matched by corresponding elevations lower than that of $\tilde{P}$, and the average must be just equal to the elevation of $\tilde{P}$, namely, the function h at P.

Wiener showed how to do this in a mathematically legitimate way, thereby bringing the term Brownian motion into the language of mathematics. In the Wiener process, as one refers to Wiener's model of Brownian motion, the distances traveled are distributed according to a Gaussian curve, just as they are in Einstein's physical model of Brownian motion. Moreover, Wiener proved that almost certainly (with a probability of 1) the path is continuous but nowhere smooth. This also fits very nicely with physical intuition. A particle in Brownian motion surely cannot jump instantaneously from one point to another, so that the path should be continuous; erratic changes in direction seem to be taking place constantly, so that one might expect the path to consist entirely of sharp corners.

Wiener's work has been continued by a long line of successors; in a sense it is the fountainhead of most modern work in random processes. One of the most fruitful outcomes of this work has been its role in the development of probabilistic potential theory. To explain the circumstances that led to this highly successful merger it is necessary to turn briefly to classical, or nonprobabilistic, potential theory.

Potential theory is the mathematics of equilibrium. It studies harmonic functions, which arise whenever one has a homogeneous medium in a state of equilibrium. Consider an elastic membrane stretched across a stiff, closed frame that is twisted into some fixed shape in space [*see illustration above*]. The configuration of such a membrane is given by the height h of each point $\tilde{P}$ on the surface of the membrane. Directly below each point $\tilde{P}$ on the membrane is a point P in the base plane, which has the coordinates x, y. Thus if the coordinates x, y are given, then h is a determined quantity; h is said to be "a function of x and y," or in more concise symbolic terms, $h = h(x, y)$.

It is physically clear, and easy to prove mathematically, that h is continuous, and moreover that it has the following simple property: If P is a point in the x, y plane, and Γ is a small circle with its center at P, then the value of h at P

(that is, the height of the membrane above P) equals the average of the values of h for all points on the circle Γ. This is called the mean-value property, and a continuous function h possessing this property is called a harmonic function. In this case it can be seen that the position of $\tilde{P}$ (the point on the membrane above P) is determined by the sum of tension forces exerted on $\tilde{P}$ by the surrounding portion of the membrane. If the membrane is in equilibrium, these forces must cancel, so that the total of nearby elevations greater than the elevation of $\tilde{P}$ must be matched by corresponding elevations lower than the elevation of $\tilde{P}$, and the average must be just equal to the elevation of $\tilde{P}$, namely h at P.

Another physical problem leading to a harmonic function is the problem of temperature equilibrium. In the theory of heat flow (which long antedates the kinetic theory of heat) it is known that in a homogeneous solid the temperature at any point P tends to fall if the average nearby temperature is lower than that at P; it tends to rise if the average nearby temperature is higher than that at P. If the body is in thermal equilibrium, so that the temperature at any given point does not change with time, then the temperature at that point must equal the average temperature over the surface of a small surrounding sphere. In other words, the temperature T is a harmonic function of the coordinates x, y, z of the point P [*see illustration below*].

The remarkable discovery that all the main problems and features of classical potential theory have a mathematical counterpart in the theory of Brownian motion was foreshadowed in 1928 by the work of Richard Courant, K. O. Friedrichs and H. Lewy in Germany. The mathematical equivalence of the two theories has been fully exploited in the past two decades by a host of mathematicians, including Joseph Doob, Gilbert Hunt and Mark Kac in the U.S., E. B. Dynkin in the U.S.S.R., P. A. Meyer in France and Shizuo Kakutani and K. Ito of Japan.

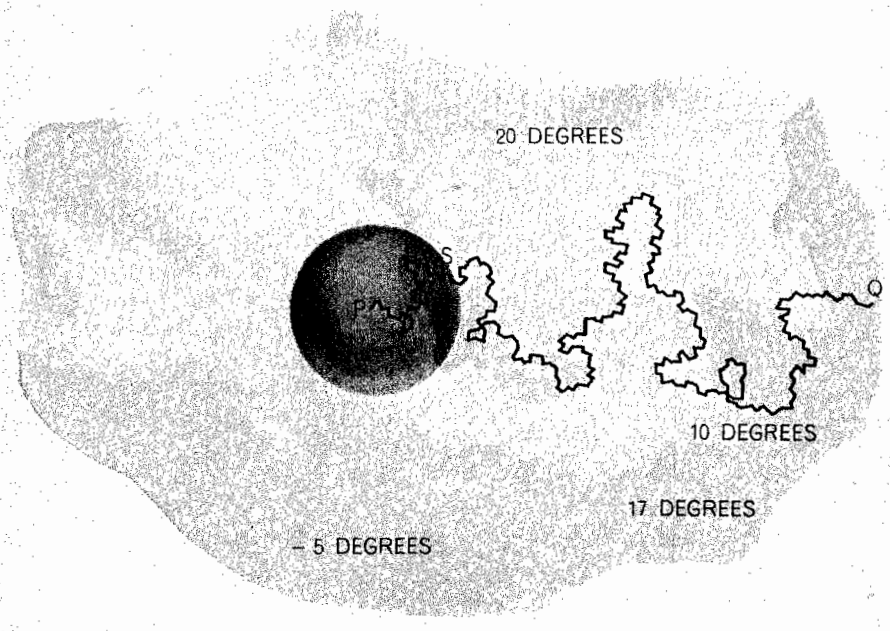
The happy result of all this work is that today any information available in one theory can be translated into a theorem in the other. In particular, it often happens that what is difficult or obscure in one theory is completely transparent in the other. We shall now give several examples to show how light can be shed in either direction by this relation.

The main connection between the theory of Brownian motion and potential theory is made by way of the central problem of potential theory, which is called the Dirichlet problem after the German mathematician P. G. L. Dirichlet. Suppose that in the foregoing example of a body at thermal equilibrium the temperature is measured at all points on the surface of the body. Some points are hot and others are cold, and if the body has been maintained in this state for a while, one can expect that a thermal equilibrium has been attained in the interior. The temperature in the interior varies from point to point, but at each fixed point it does not change with time. From these assumptions can one compute the interior temperature?

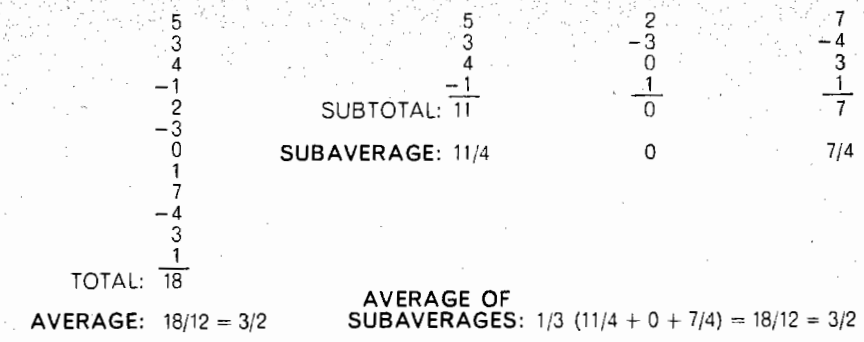
In mathematical terms what one is seeking here is a harmonic function, defined in the interior of the body, that takes on certain known values on the surface, or boundary, of the body. (In the membrane example the same mathematical problem has the following physical interpretation: Given the position of the boundary of the membrane, compute the position of the interior points.) The study of Dirichlet's problem has occupied the attention of many leading mathematical analysts of the past century. Wiener made major contributions to this study, yet he never saw how his own theory of Brownian motion could be used to solve the Dirichlet problem; this was first done by Kakutani.

To explain Kakutani's method we resort to the language of Monte Carlo or Las Vegas. Considering once again the solid body of the foregoing example, choose an interior point P at which one wants to know the equilibrium temperature. Having chosen P, one now plays a somewhat unconventional gambling game: Use the point P as the starting position for the Brownian motion of a particle. Watch the particle. It will wander around and eventually (with a probability of 1!) hit the boundary. Say it hits at a certain point Q. Then one wins an amount equal to the known temperature at Q. Of course, it is a matter of chance where the particle first hits the boundary. In short, P is a determined point in the interior, whereas Q is a random point on the boundary.

Obviously one will win at the most the maximum boundary temperature and at the least the minimum boundary temperature. Moreover, as in any gambling game, there is a certain amount one can expect to win in the long run if one plays habitually. This quantity can be found simply by playing a great many times and computing one's average winnings. We shall call this quantity the "expected value" of P, or $E(P)$ for short. The "expectation" of the game is the amount a rational gambler would be willing to pay to the gambling house for the privilege of playing. It is this quantity—the expected winnings if one starts at P—that is precisely Kakutani's solution of the Dirichlet problem.



DISTRIBUTION OF HEAT in a homogeneous solid body at thermal equilibrium is another physical problem that involves a harmonic function. Since the temperature at any given point in such a body does not change with time, the temperature at that point must equal the average temperature over the surface of a small surrounding sphere. In other words, the temperature T is a harmonic function of the coordinates x, y, z of the point P. The problem can be solved by means of the probabilistic theory of the Brownian motion of a hypothetical particle starting at P and hitting the surface of the body at a random point Q.



PRINCIPLE that the average of the subaverages equals the grand average is employed in the solution of the heat-distribution problem by means of probabilistic potential theory.

The idea is so simple that (by neglecting a few mathematical fine points) one can readily verify that $E(P)$ is indeed a solution. First of all, it should be clear that this expectation is a number. As we have defined it, it is a number of dollars, but we can equally well interpret it in degrees of temperature. Moreover, what number it is clearly depends on what starting point one designates for the particle in Brownian motion. If one starts close to a hot part of the boundary, one can expect to win more than if one starts near a cold part of the boundary. Thus a number is actually associated with each interior point P. In order to verify that $E(P)$ is the equilibrium temperature (the solution of Dirichlet's problem) there are only two criteria to check: first, that it matches the known temperature on the boundary, and second, that it is harmonic (that it is continuous and has the mean-value property) in the interior.

With respect to the first criterion, it is clear that if one starts at a point P that is actually on the boundary, then the game is over before it begins, and the payoff $E(P)$ is precisely the known temperature at the starting point; in other words, $P = Q$ with a probability of 1. Moreover, it is plausible (and can be proved rigorously) that if one starts the particle at an interior point P sufficiently close to a particular boundary point Q_0, then it is almost certain that one will first hit the boundary at a point very close to Q_0, so that the expected winnings—the equilibrium temperature assigned to P—is very close to the known temperature at Q_0. Thus Kakutani's solution does have the required boundary behavior. (In this argument it is tacitly assumed that near Q_0 the boundary is smooth and that the boundary temperature is continuous.)

Next one has to show that as a function of P the expected value $E(P)$ in this game is a harmonic function. Again the required continuity is intuitively clear. All this means is that if the starting point P is changed very slightly, then the expected winnings are changed only slightly. The payoff obviously depends on the relative distance of the starting point from the hot and cold parts of the boundary; a slight change in starting position means a slight change in these distances.

What about the mean-value property? This is the only part of the argument that is not perfectly straightforward, in that it requires the introduction of what might be called a gimmick. Draw a small sphere designated Γ around P. Now, in order to show that E is harmonic one has to show that $E(P)$ is equal to the average of all the $E(S)$'s, where S is an arbitrary point on the sphere Γ. $E(P)$ is the expected payoff of the game if the particle starts out at P. Pick a point S on Γ, and suppose that one considers only those plays of the game in which the particle first meets Γ precisely at S. Call the average winnings in these plays $E(P/S)$. Because Brownian motion has no preferred direction, each point S on Γ is equally likely to be the first, and so, by the principle that the average of the subaverages equals the grand average [*see illustration above*], one sees that $E(P)$ equals the average, taken over all points S on Γ, of $E(P/S)$.

The proof would be complete if it were now possible to show that $E(P/S)$ is the same as $E(S)$, $E(S)$ being the expected winnings for a particle starting at S, and $E(P/S)$ being the expected winnings for a particle starting at P and first meeting Γ at S. At any instant, however, the particle in Brownian motion behaves only on the basis of its present position; it is not influenced by its past. (This is called the "Markov property.") Therefore the expected behavior of a path from P through S is no different from the behavior of a path starting at S; consequently $E(S)$ does indeed equal $E(P/S)$. In physical terms this means that one could solve the equilibrium problem for heat flow or for a membrane by observations of Brownian motion, or conversely that one could find the expected outcome of a Brownian-motion experiment simply by observing the equilibrium configuration of a membrane or a heat conductor.

A noteworthy feature of the probabilistic method of solving the Dirichlet problem is that the boundary can be as irregular as one pleases. Other procedures for solving the problem all encounter complications if the surface of the domain is too "spiky" or "hairy." (The surface of a bulldog is all right, but not the surface of a Saint Bernard or a poodle.) In contrast the Brownian-motion solution is meaningful in all cases. For a badly behaved boundary the prescribed boundary values are taken "on the average," but not necessarily at each point.

Having shown how the connection between Brownian motion and potential theory has been exploited to obtain deep insights into classical potential theory by simple probabilistic arguments, we shall now give a few examples, based on the work of Kakutani and Doob, of how, on the other hand, complicated and deep questions in probability are sometimes equivalent to very simple questions in potential theory.

Our first example is the "gambler's ruin" problem. Suppose that one of us (Hersh) plays with an opponent (Griego) at matching quarters. Hersh's fortune at the start is N dollars. Griego has M dollars. Hersh resolves to play until he either "breaks the bank" (by winning M dollars) or is "ruined" (by losing N dollars).

The question is: What is the probability that Hersh will be ruined? It is intuitively clear that the answer depends on the relative sizes of M and N. If N is much less than M, ruin is very likely; if N is much greater than M, Hersh is pretty sure to break the bank. What may not be clear is that it is possible to get an exact answer!

To do this we define a second game. Imagine that you (the reader) are an onlooker at our quarter-matching. You watch our luck, and you bet a dollar that Hersh will be ruined. If Hersh is ruined, you win a dollar; otherwise you win nothing. What are your expected winnings? Clearly they are equal to Hersh's probability of being ruined. Furthermore, if we call $x(t)$ Hersh's net gain or loss at time t, then x changes from one instant to the next in the same way that a Brownian particle changes position. The Brownian particle moves to the left or the right with equal probability; Hersh's fortune increases or decreases with equal probability. The game is over when Hersh's winnings x are either $+M$ dollars or $-N$ dollars. This corresponds to

a particle in Brownian motion on the x axis between the points $x = +M$ and $x = -N$. The probability of ruin is the probability that the particle first hits the left boundary ($x = -N$) and not the right boundary ($x = M$). That means your game has a payoff equal to 1 if the particle first hits the left boundary and a payoff equal to 0 if it first hits the right boundary. This corresponds precisely to a Dirichlet problem for a one-dimensional domain (the interval between $x = -N$ and $x = M$) with boundary values 1 at $x = -N$ and 0 at $x = M$.

We could consider the corresponding equilibrium-temperature problem, but it is even simpler in this case to visualize a membrane problem. In fact, since our domain is one-dimensional (a part of the x axis) we must consider a one-dimensional elastic, say a stretched rubber band. Everyone knows that the equilibrium position of a stretched rubber band is a straight line. Since in this case the particle representing Hersh's winnings starts at $x = 0$, we are interested in the height of the rubber band above the point $x = 0$. Simple geometry shows that it is just $M/(M + N)$ [*see illustration below*]. This, then, is Hersh's probability of being ruined.

The method we have employed here is a simple and powerful one. We essentially constructed a special Dirichlet problem, taking care to choose the domain and the boundary values strategically, so that the problem would have an interesting probabilistic interpretation. The solution was available by inspection because the associated equilibrium configuration was so extremely simple.

Our next example will require a little more acquaintance with potential theory, but it will yield a much deeper probabilistic result. Choose a fixed point as an "origin," and consider the domain $\mathcal{D}$ of all points P whose distance from the origin is greater than ε and less than K. Here, as usual, ε is supposed to be a small positive number and K a very large one. In three-dimensional space $\mathcal{D}$ is the region between two concentric spheres, an inner one of radius ε and an outer one of radius K. In two dimensions the same conditions describe a ring between two concentric circles. In one dimension $\mathcal{D}$ is a pair of disconnected intervals, one to the right and one to the left of the origin [*see illustration on page 74*]. In each case we pose a Dirichlet problem by asking for a function u that is harmonic in $\mathcal{D}$ and equal to 1 on the inner surface and 0 on the outer surface. What is the solution?

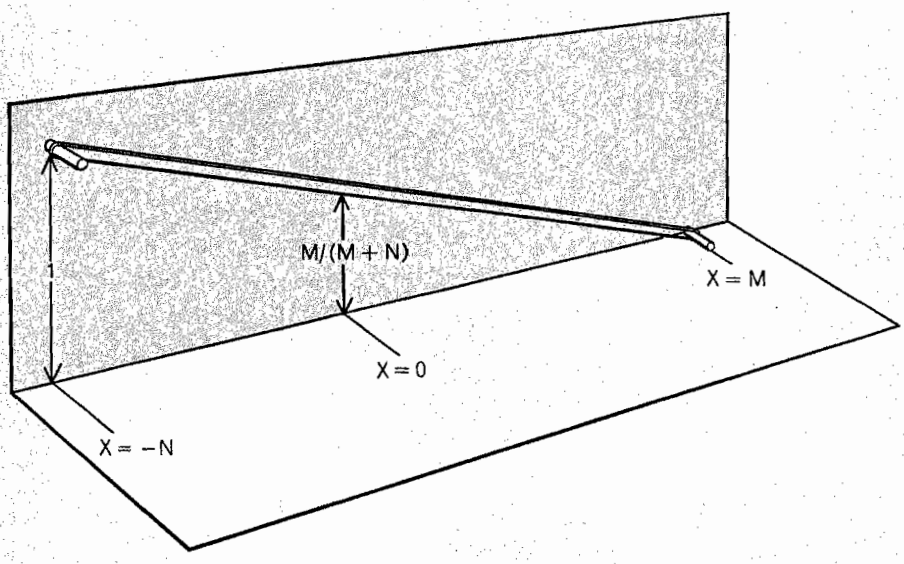
In one dimension, as in the stretched rubber band, the only harmonic functions $u(x)$ are those that have straight lines as their graphs. A comparable construction shows that in the one-dimensional case the solution to the present problem is $u(x) = (K - x)/(K - \varepsilon)$ for x greater than ε, and $u(x) = (K + x)/(K - \varepsilon)$ for x less than $-\varepsilon$.

Just as in the gambler's-ruin problem, $u(x)$ signifies the probability that if a particle starts at position x, it will hit the inner boundary (which is given a payoff equal to 1) before the outer boundary (which is given a payoff equal to 0). The formulas given above show that if K is very large, u is very close to 1. It is possible to take the limit of $u(x)$ as K goes to infinity; then $u = 1$ for all x and all ε. Since the outer boundary has now vanished to infinity, $\mathcal{D}$ is just the set of all points outside the interval from $-\varepsilon$ to $+\varepsilon$, and $u(x)$ is the probability that a particle starting at x will eventually touch that interval. Since $u = 1$, it follows that the particle is almost certain to do so. Because both the origin and the starting position x are arbitrary, the particle will arrive at every point on the line. Having arrived there, the same argument applies once more to the future, so that it will in fact almost certainly return infinitely many times to every point. This property is described by the term "recurrent," and what we have shown is that Brownian motion in one dimension is recurrent.

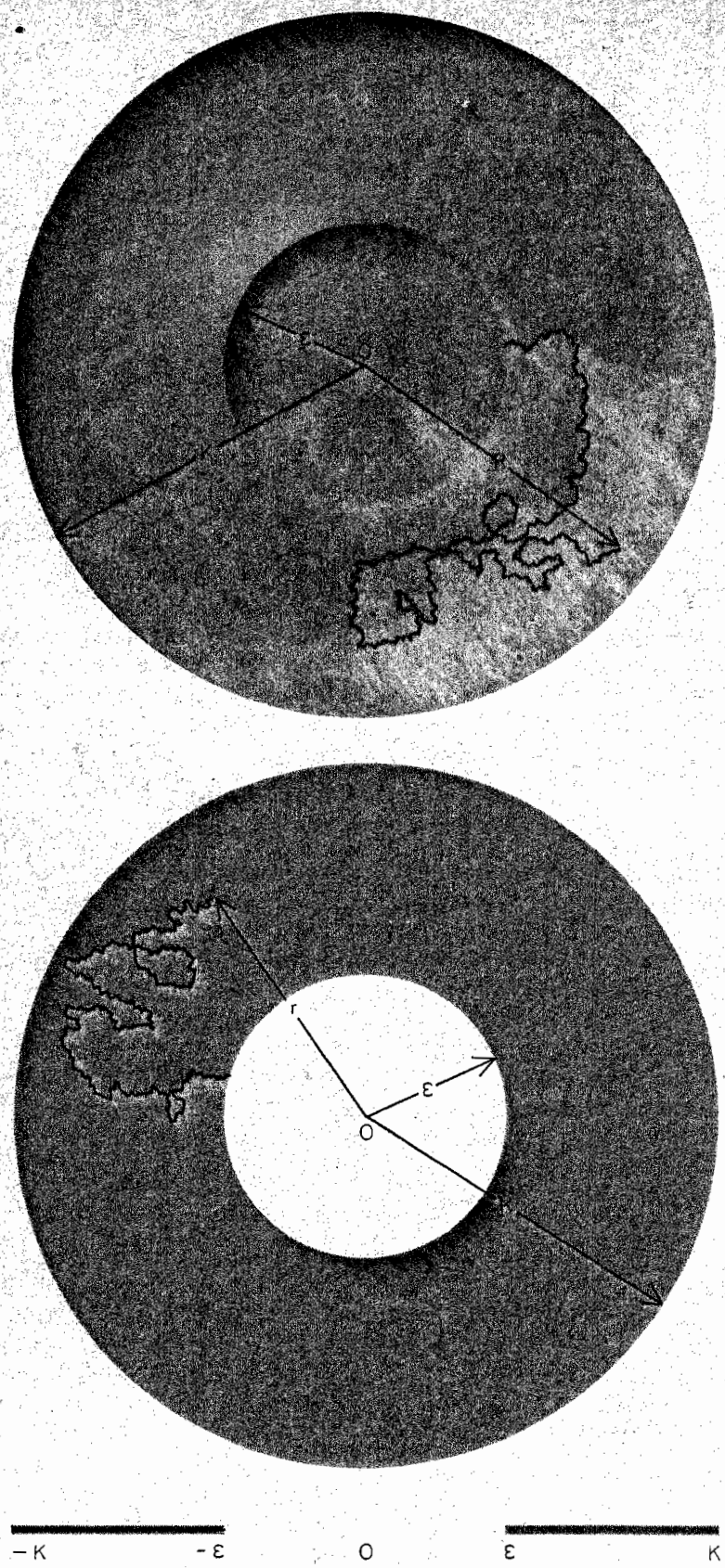
To consider the same question in two or three dimensions, one needs to know only the solutions of the corresponding two- and three-dimensional Dirichlet problems, with the boundary values 1 assigned on the inner boundary and 0 on the outer boundary. Simple considerations, which would be out of place in this article but which require only elementary calculus to carry out, show that in two dimensions the solution to our Dirichlet problem for a circular ring is $u(r) = (\log K - \log r)/(\log K - \log \varepsilon)$, where log denotes logarithm. In three dimensions, for a spherical shell, the solution is $u(R) = (1/K - 1/R)/(1/K - 1/\varepsilon)$. We use r to denote distance to the origin in the plane, and R to denote distance to the origin in three-space. In each case r and R are numbers between ε and K.

These functions $u(r)$ and $u(R)$ have the same probabilistic meaning as the $u(x)$ we just obtained in the one-dimensional case; they give the probability that a particle, starting at r or R units from the origin, will hit the inner boundary before it hits the outer one.

The interesting question is: What happens as K becomes extremely large? Since log K goes to infinity as K goes to infinity, we see that in two dimensions, as in one, Brownian motion is recurrent, that is, the particle is almost sure to re-



ONE-DIMENSIONAL ELASTIC (in this case a rubber band stretched between two pegs in a wall) is considered in finding a solution to the "gambler's ruin" problem by means of probabilistic potential theory. The game is matching quarters. One player (Hersh) has a fortune of N dollars at the start. The other player (Griego) has M dollars. Hersh resolves to play until he either "breaks the bank" (by winning M dollars) or is "ruined" (by losing N dollars). In the theory of Brownian motion the particle representing Hersh's winnings starts at $x = 0$, and the probability of ruin is the probability that the particle first hits the left boundary ($x = -N$) and not the right boundary ($x = +M$). This means that an onlooker's game has a payoff equal to 1 if the particle first hits the left boundary and a payoff equal to 0 if it first hits the right boundary. Since the equilibrium position of a stretched rubber band is a straight line, Hersh's probability of being ruined is the height of the rubber band above the point $x = 0$. Simple geometry shows that this value is just $M/(M + N)$.



DEEP RESULT in probability theory is obtained in the process of solving the following problem: Choose a fixed point as an "origin," and consider the domain $\mathcal{D}$ of all points P whose distance from the origin is greater than ε and less than K. (ε is supposed to be a very small positive number and K a very large one.) In three-dimensional space (*top*) $\mathcal{D}$ is the region between two concentric spheres, an inner one of radius ε and an outer one of radius K. In two dimensions (*middle*) the same conditions describe a ring between two concentric circles. In one dimension (*bottom*) the domain is a pair of disconnected intervals, one to the right and one to the left of the origin. In each case the problem is to find a function u that is harmonic in $\mathcal{D}$ and equal to 1 on the inner surface and 0 on the outer surface.

turn infinitely many times to a small neighborhood of any point.

In three-dimensional space, on the other hand, letting K go to infinity yields $u(R) = (-1/R)/(-1/\varepsilon) = \varepsilon/R$. This is the probability that the particle, starting at distance R greater than ε, will ever approach within ε of the origin. Since u is less than 1, there is a positive probability that the particle will wander off and never return. There is, so to speak, more room in three-space to escape. Thus Brownian motion in three dimensions is nonrecurrent. This result, which we have obtained with modest effort, is a deep result in probability theory.

The function $u(R) = \varepsilon/R$, which we have just considered, can be extended by setting it equal to 1 for R less than or equal to ε. The extended function we have defined is known as the capacitory potential of the sphere S_ε, with radius ε and center at 0. The capacitory potential of a set B is an important notion of classical potential theory; it is a function harmonic outside B, equal to 1 inside B, and equal to 0 very far from B (at infinity).

Just as in the special case of S_ε, so in very general cases the capacitory potential of B is simply the probability that a Brownian particle, starting at a given point, will ever hit B. Indeed, almost the same arguments applied to the case of a sphere would show that the probability is equal to 1 for a starting point inside B, is harmonic for a starting point outside B and is small at great distances from B.

Current work in this area has yielded far-reaching generalizations of both Brownian motion and potential theory. The interconnection between classical potential theory and Brownian motion depends heavily on the fact that Brownian motion is a Markov process, that is, its present behavior is not influenced by its past behavior. Recent investigations have shown that in a very real sense every decent Markov process corresponds to some generalized potential theory. For example, the classical theory of Riesz potentials corresponds to what are called the stable processes of probability theory. Moreover, Markov chains (which are discrete Markov processes) have their own potential theories.

Thus the probabilistic viewpoint in potential theory has unified and clarified the underlying principles of potential theory, and conversely concepts borrowed from potential theory and applied to probability theory have demonstrated the deep analytic structure of Markov processes. This has helped to end the isolation in the mathematical realm that probability theory has suffered from to some degree in the past.